SOLUTION OF NONSTEADY HEAT-CONDUCTION PROBLEMS FOR A HOLLOW CYLINDER ON AN ANALOG COMPUTER

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We discuss the use of the MN-7 analog computer to solve the differential heat-conduction equation for a hollow cylinder in the case of boundary conditions of the second and third kinds, with uniform initial temperature distribution.

The solutions in the literature [1-6] for the problems of nonsteady heat conduction for a hollow cylinder encompass the simplest cases, or those which have little practical significance. Computer techniques permit us to find solutions for the most complex of physical problems [7-9].

In [10] we find a scheme and method for the solution of the problems of nonsteady heat conduction for a hollow cylinder under boundary conditions of first kind on an MN-7 analog computer.

Below we discuss the solutions for the problems of nonsteady heat conduction for a hollow cylinder in the case of boundary conditions of the second and third kinds, as derived by means of the MN-7 analog computer.

An unbounded hollow cylinder, at the initial instant of time, exhibits a uniform temperature (t_0) through the cross section. Let us examine two cases of the solution of the differential heat-conduction equation for a cylinder with inside-to-outside radius ratios of 0.3, 0.5, and 0.7, at a constant temperature for the ambient medium: 1) between the medium and the two surfaces of the cylinder there is transfer of heat in accordance with the law of convection; 2) between the medium and the outside surface of the cylinder there is transfer of heat in accordance with the law of convection; the inside surface is ideally insulated against heat.

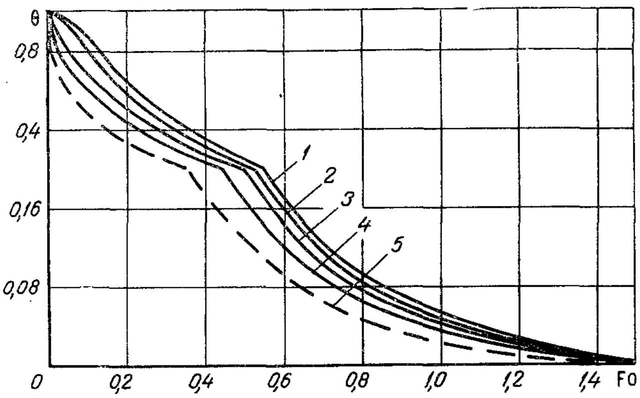

Fig. 1. Relative temperature of the hollow cylinder for K = 0.3; Bi = 2; A = 0: 1) $\theta_0 = \theta_1$; 2) θ_2; 3) θ_3; 4) θ_4; 5) θ_5.

The solution of the problem includes the following parts: a) conversion of the differential heat-conduction equation to machine form; b) solution of the problem on the computer; c) conversion of the machine solutions to the solutions of the physical problem.

We present the differential heat-conduction equation for an axisymmetric cylinder in converted form:

$$\frac{\partial\theta}{\partial \mathrm{Fo}} = \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial\theta}{\partial x}\right);$$

$$1 \geqslant \theta \geqslant 0;\ \mathrm{Fo} > 0;\ K \leqslant x \leqslant 1; \qquad (1)$$

$$\theta = \frac{t - t_{\mathrm{av}}}{t_0 - t_{\mathrm{av}}};\quad x = \frac{r}{R_{\mathrm{out}}}.$$

The initial condition is $\theta(0; x) = 1$.

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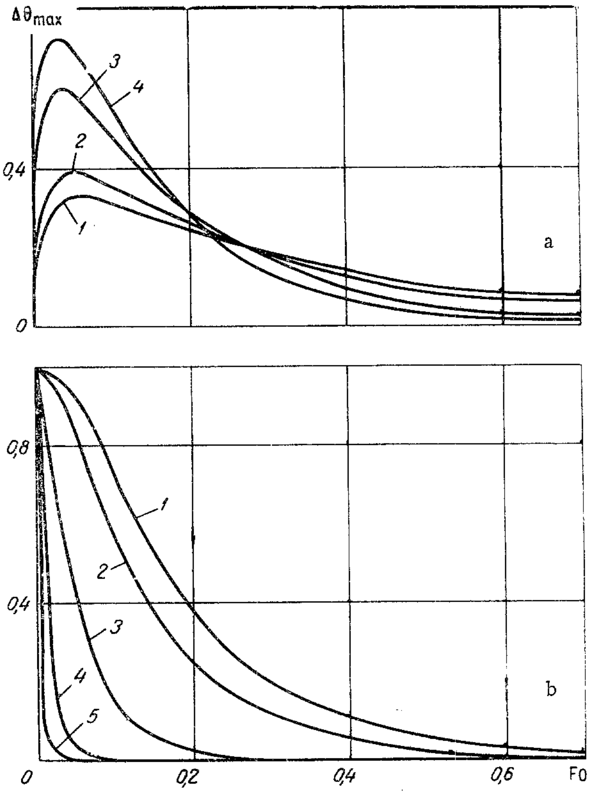

Fig. 2. Maximum temperature difference $\Delta\theta_{max} = \theta_0 - \theta_5$ in a cylinder with K = 0.3 (a) and various values of K (b) for heat transfer between the side surface on the outside and the medium: a: 1) Bi = 1; 2) 2; 3) 5; 4) 10; b: 1) K = 0.1; 2) 0.3; 3) 0.6; 4) 0.8; 5) 0.9.

The boundary conditions (for the first case):

$$\theta_{av} = 0;$$

$$\left.\frac{\partial\theta}{\partial x}\right|_{x=1} = \mathrm{Bi}\,\theta\,(\mathrm{Fo};\ 1);$$

$$\left.\frac{\partial\theta}{\partial x}\right|_{x=K} = -\mathrm{Bi}\,A\theta\,(\mathrm{Fo};\ K). \qquad (2)$$

The hollow cylinder is divided radially into z layers of identical thickness.

Using the method of finite differences, we write the relative temperature (θ) as a function of the Fo number for the surfaces of the cylindrical layers in the form of the ordinary differential equation

$$\theta_i' = \frac{1}{h^2}\left[\left(1+\frac{0,5h}{K+ih}\right)\theta_{i+1} - 2\theta_i + \left(1-\frac{0,5h}{K+ih}\right)\theta_{i-1}\right], \qquad (3)$$

where i = 1, 2, 3, . . . , z − 1.

With (3) the partial differential heat-conduction equations is replaced by a system of z − 1 ordinary differential equations.

The boundary conditions in finite differences are written as follows:

$$\frac{\theta_{z-1}-\theta_z}{\Delta x} = \mathrm{Bi}\,\theta_z \quad \text{or} \quad \theta_z = \frac{\theta_{z-1}}{1+\mathrm{Bi}\,h};$$

$$\frac{\theta_1-\theta_0}{\Delta x} = \mathrm{Bi}\,A\theta_0 \quad \text{or} \quad \theta_0 = \frac{\theta_1}{1+\mathrm{Bi}\,A\,h}. \qquad (4)$$

With z = 5 and boundary conditions (4) we compiled the following system of ordinary differential equations:

$$\begin{aligned}
\theta_1' &= -a_{11}\theta_1 + a_{12}\theta_2;\\
\theta_2' &= a_{21}\theta_1 - a_{22}\theta_2 + a_{23}\theta_3;\\
\theta_3' &= a_{32}\theta_2 - a_{33}\theta_3 + a_{34}\theta_4;\\
\theta_4' &= a_{43}\theta_3 - a_{44}\theta_4.
\end{aligned} \qquad (5)$$

The computer systems of equations have been derived in the following form:

$$\begin{aligned}
U_1' &= -k_{11}U_1 + k_{12}U_2;\\
U_3' &= k_{21}U_1 - k_{22}U_2 + k_{23}U_3;\\
U_3' &= k_{32}U_2 - k_{33}U_3 + k_{34}U_4;\\
U_4' &= k_{43}U_3 - k_{44}U_4.
\end{aligned} \qquad (6)$$

The numerical values of the constants in the computer systems of equations for the nonsteady heat-conduction cases under consideration are given in Table 1.

The solutions of the systems of equations (5) derived after conversion of the solutions of the computer systems of equations (6) are shown graphically. As an example, Fig. 1 shows the curves for a hollow cylinder: the dashed line shows the approximate values of the relative temperature for the side surfaces, these having been calculated in accordance with the boundary conditions.

Figure 2 shows the maximum temperature difference across the cylinder, determined by means of the curves in Fig. 1.

With boundary conditions of the second kind we solved the differential equation (1) for the cylinder, and in this equation:

TABLE 1. Constants in the Equations of the Computer Systems and the Scales for the Representation of Relative Time

Boundary conditions	K	Bi	A	k_{11}	k_{12}	k_{21}	k_{22}	k_{23}	k_{32}	k_{33}	k_{34}	k_{43}	k_{44}	M_{Fo}
$\theta_0=\frac{\theta_1}{1+\mathrm{Bi}Ah}$; $\theta_5=\frac{\theta_4}{1+\mathrm{Bi}h}$	0,3	10	1	0,420	0,295	0,225	0,510	0,285	0,230	0,510	0,280	0,235	0,395	0,005
	0,5	10	1	0,154	0,108	0,093	0,200	0,107	0,094	0,200	0,106	0,094	0,147	0,001
	0,7	10	1	0,456	0,289	0,268	0,556	0,288	0,268	0,556	0,288	0,269	0,448	0,001
	0,3	5	1	0,385	0,295	0,225	0,510	0,285	0,230	0,510	0,280	0,235	0,350	0,005
	0,5	5	1	0,556	0,432	0,372	0,800	0,428	0,376	0,800	0,424	0,376	0,516	0,004
	0,7	5	1	0,350	0,289	0,268	0,556	0,288	0,268	0,556	0,288	0,269	0,335	0,001
	0,3	2	1	0,342	0,295	0,225	0,510	0,285	0,230	0,510	0,280	0,235	0,295	0,005
	0,5	2	1	0,496	0,432	0,372	0,800	0,428	0,376	0,800	0,424	0,376	0,448	0,004
	0,7	2	1	0,318	0,289	0,268	0,556	0,288	0,268	0,556	0,288	0,269	0,300	0,001
	0,3	1	1	0,330	0,295	0,225	0,510	0,285	0,230	0,510	0,280	0,235	0,282	0,005
	0,5	1	1	0,468	0,432	0,372	0,800	0,428	0,376	0,800	0,424	0,276	0,416	0,004
	0,7	1	1	0,304	0,289	0,268	0,556	0,288	0,368	0,556	0,288	0,269	0,286	0,001
$\theta_0=\theta_1$; $\theta_5=\frac{\theta_4}{1+\mathrm{Bi}h}$	0,3	10	0	0,295	0,295	0,225	0,510	0,285	0,230	0,510	0,280	0,235	0,395	0,005
	0,5	10	0	0,108	0,108	0,093	0,200	0,107	0,094	0,200	0,106	0,094	0,147	0,001
	0,7	10	0	0,289	0,289	0,268	0,556	0,288	0,268	0,556	0,288	0,269	0,448	0,001
	0,3	5	0	0,295	0,295	0,225	0,510	0,285	0,230	0,510	0,280	0,235	0,350	0,005
	0,5	5	0	0,432	0,432	0,372	0,800	0,428	0,376	0,800	0,424	0,376	0,516	0,004
	0,7	5	0	0,289	0,289	0,268	0,556	0,288	0,268	0,556	0,288	0,269	0,335	0,001
	0,3	2	0	0,295	0,295	0,225	0,510	0,285	0,230	0,510	0,280	0,235	0,295	0,005
	0,5	2	0	0,432	0,432	0,372	0,800	0,428	0,376	0,800	0,424	0,376	0,448	0,004
	0,7	2	0	0,289	0,289	0,268	0,556	0,288	0,268	0,556	0,288	0,269	0,300	0,001
	0,3	1	0	0,295	0,295	0,225	0,510	0,285	0,230	0,510	0,280	0,235	0,282	0,005
	0,5	1	0	0,432	0,432	0,372	0,800	0,428	0,376	0,800	0,424	0,376	0,416	0,004
	0,7	1	0	0,289	0,289	0,268	0,556	0,288	0,268	0,556	0,288	0,269	0,286	0,001

TABLE 2. Constants in the Equations of the Computer Systems and the Scales for the Representation of Relative Time

Boundary conditions	K	ω	M_{Fo}	$\frac{q_{out}R_{out}}{\lambda(t_{max}-t_{min})}$	k_{11}	k_{12}	k_x	k_{21}
$\theta_0=\theta_1-\frac{q_{out}R_{out}h\omega}{\lambda(t_{max}-t_{min})}$; $\theta_5=\theta_4-\frac{q_{out}R_{out}h}{\lambda(t_{max}-t_{min})}$	0,3	1	0,005	1	0,295	0,295	0,300	0,225
	0,5	1	0,004	1	0,432	0,432	0,367	0,372
	0,7	1	0,001	1	0,289	0,289	0,160	0,268
$\theta_0=\theta_1$; $\theta_5=\theta_4-\frac{q_{out}R_{out}h}{\lambda(t_{max}-t_{min})}$	0,3	0	0,005	1	0,295	0,295	0	0,225
	0,5	0	0,004	1	0,432	0,432	0	0,372
	0,7	0	0,001	1	0,289	0,289	0	0,268

Boundary conditions	k_{22}	k_{23}	k_{32}	k_{33}	k_{34}	k_{43}	k_{44}	k_y
$\theta_0=\theta_1-\frac{q_{out}R_{out}h\omega}{\lambda(t_{max}-t_{min})}$; $\theta_5=\theta_4-\frac{q_{out}R_{out}h}{\lambda(t_{max}-t_{min})}$	0,510	0,285	0,230	0,510	0,280	0,235	0,285	0,387
	0,800	0,428	0,376	0,800	0,424	0,376	0,426	0,422
	0,556	0,288	0,268	0,556	0,288	0,269	0,289	0,172
$\theta_0=\theta_1$; $\theta_5=\theta_4-\frac{q_{out}R_{out}h}{\lambda(t_{max}-t_{min})}$	0,510	0,285	0,230	0,510	0,280	0,235	0,285	0,387
	0,800	0,428	0,376	0,800	0,424	0,376	0,426	0,422
	0,556	0,288	0,268	0,556	0,288	0,269	0,289	0,172

$$\theta=\frac{t-t_{min}}{t_{max}-t_{min}}.$$

The boundary conditions (for the first case)

$$\left.\frac{\partial\theta}{\partial x}\right|_{x=l}=-\frac{q_{out}R_{out}}{\lambda(t_{max}-t_{min})};\qquad \left.\frac{\partial\theta}{\partial x}\right|_{x=K}=\frac{q_{out}R_{out}\omega}{\lambda(t_{max}-t_{min})}. \tag{7}$$

With the finite differences the boundary conditions were converted to the form (z = 5)

$$\frac{\theta_4-\theta_5}{h}=\frac{q_{out}R_{out}}{\lambda(t_{max}-t_{min})}\quad\text{or}\quad\theta_5=\theta_4-\frac{q_{out}R_{out}h}{\lambda(t_{max}-t_{min})};$$
$$\frac{\theta_1-\theta_0}{h}=\frac{q_{out}R_{out}\omega}{\lambda(t_{max}-t_{min})}\quad\text{or}\quad\theta_0=\theta_1-\frac{q_{out}R_{out}h\omega}{\lambda(t_{max}-t_{min})}. \tag{8}$$

For the boundary conditions (8) we derived a system of differential equations of the following form:

$$\theta_1'=-a_{11}\theta_1+a_{12}\theta_2-b_{13}\frac{q_{out}R_{out}h\omega}{\lambda(t_{max}-t_{min})};$$
$$\theta_2'=a_{21}\theta_1-a_{22}\theta_2+a_{23}\theta_3;$$
$$\theta_3'=a_{32}\theta_2-a_{33}\theta_3+a_{34}\theta_4; \tag{9}$$
$$\theta_4'=a_{43}\theta_3-a_{44}\theta_4-b_{45}\frac{q_{out}R_{out}h}{\lambda(t_{max}-t_{min})}.$$

The computer systems of equations corresponding to systems (9) are the following:

$$U_1'=-k_{11}U_1+k_{12}U_2-k_xU_y;\quad U_2'=k_{21}U_1-k_{22}U_2+k_{23}U_3;$$
$$U_3'=k_{32}U_2-k_{33}U_3+k_{34}U_4;\quad U_4'=k_{43}U_3-k_{44}U_4-k_yU_y. \tag{10}$$

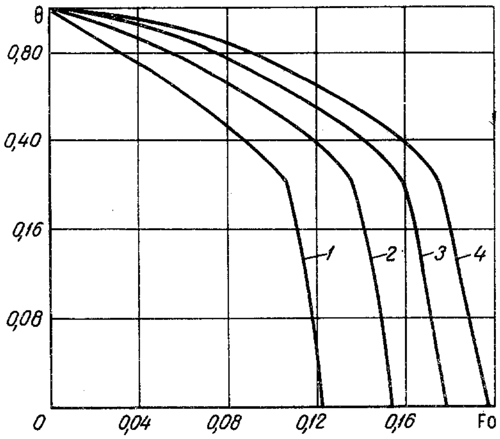

Fig. 3. Relative temperature of the hollow cylinder for K = 0.3, $(q_{out}R_{out})/\lambda(t_{max} - t_{min}) = 1$; $\omega = 0$: 1) $\theta_0 = \theta_1$; 2) θ_2; 3) θ_3; 4) θ_4.

The schemes for the set of problems relating to a hollow cylinder for boundary conditions of the third and second kinds differ from each other only in the additional resistances and voltages simulating the quantities

$$b_{13}\,\frac{q_{out}R_{out}h\omega}{\lambda\,(t_{max}-t_{min})} \quad \text{or}\ b_{45}\,\frac{q_{out}R_{out}h}{\lambda\,(t_{max}-t_{min})}\;.$$

The numerical values of the constants in the computer systems of equations for the cases of nonsteady heat conduction examined here are shown in Table 2.

For $k_{11} \leq k_{12}$ the systems of equations proved to be unstable. By means of a ν-transformation we derived stable systems. The coefficients with identical subscripts were ascribed new values as a result:

$$k_{11}^{*}=k_{11}+\nu= \ 0{,}345;\ 0{,}482;\ 0{,}309;\ 0{,}345;\ 0{,}482;\ 0{,}309;$$

$$k_{22}^{*}=k_{33}^{*}=k_{22}+\nu=k_{33}+\nu= \ 0{,}560;\ 0{,}850;\ 0{,}576;\ 0{,}560;\ 0{,}850;\ 0{,}576.$$

The following sequence was adopted for the solutions of the problems: initially we solved the converted machine systems of equations; the solutions of these equations were converted into solutions of the computer systems (10), and the latter were converted into solutions for the systems of equations (9).

The initial voltages (U_{in}) and the voltages simulating the free terms (U_y) were assumed to be equal to 50 V.

Figure 3 shows the curves for certain cases of nonsteady heat conduction in the case of boundary conditions of the second kind.

NOTATION

t	is the temperature, °C;
t_0	is the initial temperature of the cylinder;
t_{av}	is the average temperature;
r	is the instantaneous cylinder radius;
R_{out}	is the outside cylinder radius;
$Fo = a\tau/R_{out}^2$	is the Fourier number;
a	is the coefficient of thermal diffusivity;
τ	is the time;
$K = R_{in}/R_{out}$	is the cylinder radius ratio;
R_{in}	is the inside radius;
$Bi = \alpha_{out}R_{out}/\lambda$	is the Biot number;
α_{out}	is the coefficient of heat transfer between the outside surface of the cylinder and the medium;
λ	is the coefficient of thermal conductivity;
$A = \alpha_{in}/\alpha_{out}$	is a relative quantity;
α_{in}	is the coefficient of heat transfer between the inside surface of the cylinder and the medium;
$\theta_i' = d\theta_i/d\,Fo$;	
$h = (1 - K)/z$	is the relative thickness of the cylindrical layer;
$k_{11}, k_{12}, \ldots, k_{44}$	are the constants for the computer systems of equations;
$k_{ij} = M_{Fo}\,a_{ij}$;	
a_{ij}	are the constants in the systems of ordinary differential equations;
$M_{Fo} = Fo/t_c$	is the scale for the representation of the relative time;
t_c	is the computer time, sec;
$\omega = q_{in}/q_{out}$	is the ratio of the heat-flux density at the inside surface of the cylinder (q_{in}) to the heat-flux density at the outside surface on the side (q_{out});

t_{min}, t_{max}	are the finite minimum and initial maximum temperature, uniform through the cross section of the cylinder;
$k_x = b_{13}h\omega(M_{Fo}M_y/M_\theta), k_y = b_{45}h(M_{Fo}M_y/M_\theta)$	are constants in the computer systems (10);
$M_\theta = \theta/U$	is the scale for the representation of the relative temperature, 1/V;
$M_y = (q_{out}R_{out}/\lambda(t_{max} - t_{min}))(1/U_y)$	is the scale for the representation of $q_{out}R_{out}/\lambda(t_{max} - t_{min})$, 1/V.

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